2. Discrete Linear Stochastic Processes/Models

2.1. Moving average processes

- **Definition:**
  A random sequence $X(n)$ is a moving average process of order $q$ (MA($q$)) if for any $n$:
  \[ X(n) = Z(n) + \sum_{i=1}^{q} \theta_i Z(n-i) \]
  where $Z(n)$ is a white Gaussian process.

- **Transversal filter implementation of a MA($q$) process:**

- **Impulse response of the transversal filter:**

- **Stability and causality:**
  Transversal filters are stable and causal.

- **Transfer function of the transversal filter:**

  \[ H(f) = 1 + \sum_{i=1}^{q} \theta_i \exp(-j2\pi f) \]

  **Proof:**
  \[
  x(n) = \sum_{i=1}^{q} \theta_i z(n-i) + z(n) \\
  X(f) = \sum_{i=1}^{q} \theta_i \exp(-j2\pi f) Z(f) + Z(f) \\
  = \left[ 1 + \sum_{i=1}^{q} \theta_i \exp(-j2\pi f) \right] Z(f)
  \]

- **Power spectrum of a MA($q$) process:**

  \[ S_{XX}(f) = \left| 1 + \sum_{i=1}^{q} \theta_i \exp(-j2\pi f) \right|^2 \sigma_Z^2 \]

- **Mean value and autocorrelation function of a MA($q$) process:**

  \[ \mu_X = 0 \]
  \[ R_{XX}(k) = \sigma_Z^2 \delta_{kq} \]

- **Example: MA(1)**

  \[ X(n) = Z(n) + Z(n-1) \quad (\theta_1 = 1) \]
- Impulse response and autocorrelation function of the transversal filter

\[ h(n) = \begin{cases} 
1 & n \in \{0, 1\} \\
0 & \text{elsewhere}
\end{cases} \]

\[ R_{hh}(k) = \begin{cases} 
2 & k \in \{0\} \\
1 & k \in \{-1, 1\} \\
0 & \text{elsewhere}
\end{cases} \]

- Transfer function:

\[ H(f) = 1 + \exp(-j2\pi f) \quad |f| \leq 0.5 \]

\[ = \exp(-j\pi f)\exp(j\pi f) + \exp(-j\pi f) \]

\[ = 2\exp(-j\pi f)\cos(\pi f) \]

\[ |H(f)| = 2\cos(\pi f) \]

\(|f| \leq 0.5\)

- Autocorrelation function of \(X(n)\):

\[ R_{XX}(k) = \sigma_Z^2 R_{hh}(k) \]

\[ = \begin{cases} 
2\sigma_Z^2 & k \in \{0\} \\
\sigma_Z^2 & k \in \{-1, 1\} \\
0 & \text{elsewhere}
\end{cases} \]

- Power spectrum of \(X(n)\):

\[ S_{XX}(f) = \sigma_Z^2|H(f)|^2 \]

\[ = 4\sigma_Z^2\cos^2(\pi f) \]

\(2.2.\) Autoregressive processes

- Definition:

A random sequence \(X(n)\) is an autoregressive process of order \(p\) (AR\((p)\)) if it is WSS and for any \(n\):

\[ X(n) = \sum_{i=1}^{p} \phi_i X(n-i) + Z(n) \]

where \(Z(n)\) is a white Gaussian process.
• Recursive filter implementation:

An AR(\(p\)) process \(X(n)\) is called causal and stable if there exists an infinite causal and stable transversal filter with impulse response \(h(n)\) such that

\[
X(n) = \sum_{i=0}^{\infty} h(i)Z(n-i)
\]

\[= h(n)Z(n)\]

Let us define the polynomial

\[
\phi(z) = 1 - \sum_{i=1}^{p} \phi_i z^{-i}
\]

\(z\) : complex variable.

Then, the AR process \(X(n)\) is causal and stable, if, and only if, the roots of \(\phi(z)\) are located inside the unit circle, i.e. if \(\phi(z)\) factorizes according to

\[
\phi(z) = \prod_{i=1}^{p} (1 - \rho_i z^{-i})
\]

with \(|\rho_i| < 1, i = 1, \ldots, p\).

Location of the roots of \(\phi(z)\) in the complex plane when \(X(n)\) is causal and stable:

The impulse response of a causal and stable AR(\(p\)) process is determined by the identity

\[
\sum_{i=0}^{\infty} h(i)z^{-i} = \frac{1}{\phi(z)} \quad |z| \geq 1
\]

• Causal and stable AR processes:

• Transfer function of the recursive filter:

\[
H(f) = \frac{1}{1 - \sum_{i=1}^{p} \phi_i \exp(-j2\pi if)}
\]

Proof:

\[
x(n) = \sum_{i=1}^{p} \phi_i x(n-i) + z(n)
\]

\[
X(f) = \sum_{i=1}^{p} \phi_i \exp(-j2\pi if)X(f) + Z(f)
\]

\[
= \left[\sum_{i=1}^{p} \phi_i \exp(-j2\pi if)\right]X(f) + Z(f)
\]

• Power spectrum of an AR(\(p\)) process:

\[
S_{XX}(f) = \frac{\sigma_z^2}{1 - \sum_{i=1}^{p} \phi_i \exp(-j2\pi if)}
\]
• Mean value and autocorrelation function of a causal AR(p) process:
  If the AR process $X(n)$ is causal, then
  \[ \mu_X = 0 \]
  \[ R_{XX}(k) = \sigma_Z^2 R_{hh}(k) \]

• Example: AR(1):
The first-order recursive filter discussed in the previous chapter with a white Gaussian process as the input signal generates an AR(1) process.

• Yule-Walker equations:
Let $X(n)$ be a causal AR(p) process. Then, for $k = 0, \ldots, p$, we have
\[ X(n) = \sum_{i=1}^{p} \phi_i X(n-i) + Z(n) \]
\[ X(n) X(n-k) = \sum_{i=1}^{p} \phi_i X(n-i) X(n-k) + Z(n) X(n-k) \]
\[ \mathbb{E}[X(n) X(n-k)] = \sum_{i=1}^{p} \phi_i \mathbb{E}[X(n-i) X(n-k)] + \mathbb{E}[Z(n) X(n-k)] \]
\[ R_{XX}(n,n-k) = \sum_{i=1}^{p} \phi_i R_{XX}(n-i,n-k) + \sigma_Z^2 \delta(k) \]
\[ R_{XX}(k) = R_{XX}(-k) = \sum_{i=1}^{p} \phi_i R_{XX}(i-k) + \sigma_Z^2 \delta(k) \]

Using a vector notation, for $0 \leq k \leq p$
\[ R_{XX}(k) = [R_{XX}(1-k), \ldots, R_{XX}(p-k)] \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_p \end{bmatrix} + \sigma_Z^2 \delta(k) \quad (2.1) \]

For $k > p$:
\[ R_{XX}(k) = [R_{XX}(k-1), \ldots, R_{XX}(k-p)] \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_p \end{bmatrix} \quad (2.2) \]

Let us define
\[ \Phi = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_p \end{bmatrix}, \quad \gamma = \begin{bmatrix} R_{XX}(1) \\ \vdots \\ R_{XX}(p) \end{bmatrix} \]
\[ \Gamma = \begin{bmatrix} R_{XX}(0) & R_{XX}(1) & \ldots & R_{XX}(p-1) \\ R_{XX}(-1) & R_{XX}(0) & \ldots & R_{XX}(p-2) \\ \vdots & \vdots & \ddots & \vdots \\ R_{XX}(-(p-1)) & R_{XX}(-(p-2)) & \ldots & R_{XX}(0) \end{bmatrix} \]

Note that $\Gamma$ is symmetric.

Then, for $k = 0$ Identity (2.1) becomes
\[ R_{XX}(0) = \gamma^{T} \Phi + \sigma_Z^2 \]

Inserting $k = 1, \ldots, p$ in (2.1) yields $p$ identities that can be concatenated in a matrix form according to
\[ \begin{bmatrix} R_{XX}(1) \\ \vdots \\ R_{XX}(p) \end{bmatrix} = \begin{bmatrix} R_{XX}(0) & R_{XX}(1) & \ldots & R_{XX}(p-1) \\ R_{XX}(-1) & R_{XX}(0) & \ldots & R_{XX}(p-2) \\ \vdots & \vdots & \ddots & \vdots \\ R_{XX}(-(p-1)) & R_{XX}(-(p-2)) & \ldots & R_{XX}(0) \end{bmatrix} \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_p \end{bmatrix} \]

\[ \gamma = \Gamma \Phi \]

Comments:
• The feed-back coefficients $\phi_1, \ldots, \phi_p$ of the recursive filter and the variance $\sigma_Z^2$ of the white Gaussian input process $Z(n)$ can be computed from $R_{XX}(0), \ldots, R_{XX}(p)$ via the Yule-Walker equations and vice-versa.
• The samples $R_{XX}(k), k > p$ can be recursively computed from $\phi_1, \ldots, \phi_p$ and $R_{XX}(k-1), \ldots, R_{XX}(k-p)$ by using Identity (2.2).
2.3. Autoregressive moving average processes

- **Definition:**
  A random sequence $X(n)$ is an autoregressive moving average process (ARMA($p,q$)) if it is WSS and for any $n$:
  \[ X(n) = \sum_{i=1}^{p} \theta_i X(n-i) + \sum_{i=1}^{q} \theta_i Z(n-i) + Z(n) \]
  where $Z(n)$ is a white Gaussian process.

- **Filter implementation:**
  \[
  X(n) = \sum_{i=1}^{p} \theta_i X(n-i) + \sum_{i=1}^{q} \theta_i Z(n-i) + Z(n)
  \]
  \[
  h(n) = \sum_{i=0}^{\infty} h_i z^{-i}
  \]
  \[
  S_{XX}(f) = \frac{1 + \sum_{i=1}^{q} \theta_i \exp(-j2\pi if)}{1 - \sum_{i=1}^{p} \theta_i \exp(-j2\pi if)} \sigma_Z^2
  \]

- **Causal and stable ARMA processes:**
  An ARMA($p,q$) process $X(n)$ is called causal and stable if there exists an infinite causal and stable transversal filter with impulse response $h(n)$ such that
  \[ X(n) = \sum_{i=0}^{\infty} h(i)Z(n-i) = h(n)z^n Z(n) \]
  Let us define
  \[
  \theta(z) = 1 + \sum_{i=1}^{q} \theta_i z^{-i} \quad \text{and} \quad \phi(z) = 1 - \sum_{i=1}^{p} \theta_i z^{-i}
  \]
  A necessary and sufficient condition for an ARMA($p,q$) process to be causal and stable is that the polynomial $\phi(z)$ has no roots inside the unit circle.
  The impulse response of a causal and stable ARMA($p,q$) process is then determined by the identity
  \[
  \sum_{i=0}^{\infty} h_i z^{-i} = \frac{\theta(z)}{\phi(z)} \quad |z| \geq 1
  \]
  In the above considerations we assume that $\theta(z)$ and $\phi(z)$ have no common root.

- **Transfer function of the filter:**
  \[
  H(f) = \frac{1 + \sum_{i=1}^{q} \theta_i \exp(-j2\pi if)}{1 - \sum_{i=1}^{p} \theta_i \exp(-j2\pi if)}
  \]
  *Proof:* Similar as before.

- **Power spectrum of an ARMA($p,q$) process:**
  \[
  S_{XX}(f) = \frac{1 + \sum_{i=1}^{q} \theta_i \exp(-j2\pi if)}{1 - \sum_{i=1}^{p} \theta_i \exp(-j2\pi if)} \sigma_Z^2
  \]
• **Mean value and autocorrelation function of a causal ARMA(p,q) process:**
  If the ARMA process $X(n)$ is causal,
  \[
  \mu_X = 0 \\
  R_{XX}(k) = \sigma_Z^2 R_{hh}(k)
  \]

• **Importance of ARMA(p,q) processes:**
  - Because of the linearity property of ARMA(p,q) processes, analytical expressions can be derived which describe their statistical behavior, i.e. their autocorrelation and power spectrum.
  - For any given zero-mean WSS process $Y(n)$ with autocorrelation function $R_{YY}(k)$ there exists an ARMA(p,q) process $X(n)$ such that
    \[
    R_{YY}(k) = R_{XX}(k) \quad |k| \leq K.
    \]
  In this sense, any WSS process can be approximated by an ARMA(p,q) process.