Problem 10.1: Transformation for trend elimination

Solution

a) Transform $Y(n)$ into a WSS process

Let $\Delta$ denote the 1st order discrete derivative:

$$\Delta Y(n) = Y(n) - Y(n-1)$$

$$= an^2 + V(n) - a(n-1)^2 - V(n-1)$$

$$= an^2 - a(n-1)^2 + V(n) - V(n-1)$$

$$= a[n^2 - (n-1)^2] + V(n) - V(n-1)$$

$$= a\left(2n - 1\right) + V(n) - V(n-1).$$

Applying the 2nd order derivative yields

$$\Delta^2 Y(n) = \Delta(\Delta Y(n))$$

$$= \Delta Y(n) - \Delta Y(n-1)$$

$$= a(2n - 1) + V(n) - V(n-1) - a\left[2(n-1) - 1\right] - V(n-1) + V(n-2)$$

$$= 2an - a - 2an + 3a + V(n) - 2V(n-1) + V(n-2)$$

$$= 2a + V(n) - 2V(n-1) + V(n-2).$$

Define

$$X(n) \doteq \Delta^2 Y(n)$$

$$= 2a + V(n) - 2V(n-1) + V(n-2).$$

As a linear combination of the zero-mean WSS process $V(n - k), k = 0, 1, 2, 3$ and the constant value $2a$, $X(n)$ is a WSS process with mean $2a$. Therefore the 2nd order discrete derivative ($\Delta^2$) is the required transformation.

b) Autocorrelation function of $X(n)$:

Let

$$X(n) = 2a + Z(n),$$

where

$$Z(n) \doteq V(n) - 2V(n-1) + V(n-2).$$  \hspace{1cm} (1)

$Z(n)$ is a zero-mean WSS process. The autocorrelation function of $R_{XX}(k)$ can be
written as

\[ R_{XX}(k) = E[X(n)X(n+k)] = E[(2a + Z(n))(2a + Z(n+k))] \]
\[ = E[(2a)^2 + Z(n) \cdot 2a + Z(n+k) \cdot 2a + Z(n)Z(n+k)] \]
\[ = (2a)^2 + 2a \cdot E[Z(n)] + 2a \cdot E[Z(n+k)] + R_{ZZ}(k) \]
\[ = 4a^2 + R_{ZZ}(k). \]  

(2)

According to (1), \( Z(n) \) is obtained by passing \( V(n) \) through the following filter

![Filter Diagram](image)

Then

\[ R_{ZZ}(k) = R_{VV}(k) \ast R_{hh}(k) \]

where \( R_{hh}(k) \) is the autocorrelation function of the impulse response \( h(n) \) of the filter.

\[ h(n) = \delta(n) - 2\delta(n-1) + \delta(n-2) \]

Hence,

\[ R_{hh}(k) = \sum_{n=-\infty}^{\infty} h(n)h(n+k) \]
\[ = h(k) \ast h(-k) \]
\[ = [\delta(k) - 2\delta(k-1) + \delta(k-2)] \ast [\delta(-k) - 2\delta(-k-1) + \delta(-k-2)] \]
\[ = \delta(k+2) - 4\delta(k+1) + 6\delta(k) - 4\delta(k-1) + \delta(k-2). \]

Notice that in the above expression, \( \delta(n-k_1) \ast \delta(n-k_2) = \delta(n-(k_1+k_2)) \).
Then

\[ R_{ZZ}(k) = R_{VV}(k) \ast R_{hh}(k) \]
\[ = R_{VV}(k) \ast \left[ \delta(k + 2) - 4\delta(k + 1) + 6\delta(k) - 4\delta(k - 1) + \delta(k - 2) \right] \]
\[ = R_{VV}(k + 2) - 4R_{VV}(k + 1) + 6R_{VV}(k) - 4R_{VV}(k - 1) + R_{VV}(k - 2). \]

Inserting the above expression into (2) yields

\[ R_{XX}(k) = 4a^2 + R_{ZZ}(k) \]
\[ = 4a^2 + R_{VV}(k + 2) - 4R_{VV}(k + 1) + 6R_{VV}(k) - 4R_{VV}(k - 1) + R_{VV}(k - 2). \]

So \( R_{XX}(k) \) is \( R_{ZZ}(k) \) shifted by \( 4a^2 \) upwards.

**Problem 10.2: the sunspot data**

The m file for finding the ARMA/AR coefficients can be downloaded from the course webpage.

From figure 1 and figure 2 we can see that the frequency resolution of ARMA(9,1) model is higher than the AR(3) model.
Figure 1:
Figure 2: