Correction to “Packetized Predictive Control of Stochastic Systems over Bit-Rate Limited Channels with Packet Loss”

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Abstract—We correct the results in Section V of the above mentioned manuscript.

In [1], we showed that a particular class of networked control system (NCS) with quantization, i.i.d. dropouts and disturbances can be described as a Markov jump linear system of the form

\[ \theta_{k+1} = \bar{A}(d_k)\theta_k + \bar{B}(d_k)\nu_k, \]

where \[ \bar{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \]
and \( \{d_k\}_{k \in \mathbb{N}_0} \) is a Bernoulli dropout process, with

\[ \text{Prob}(d_k = 1) = p \in (0, 1). \]

Throughout [1] we showed that properties of the NCS can be conveniently stated in terms of the expected system matrices

\[ A(p) = \mathbb{E}(\bar{A}(d_k)) \]
and

\[ B(p) = \mathbb{E}(\bar{B}(d_k)) = [B_w \ B_n(p)], \]

and the matrix \( \bar{A} = \bar{A}(1) - \bar{A}(0). \) Unfortunately, Theorem 4 in Section V-A of [1] is incorrect. For white disturbances \( \{w_k\}_{k \in \mathbb{N}_0} \), the statement should be as given below. Non-white \( \{w_k\}_{k \in \mathbb{N}_0} \) can be accommodated by using standard state augmentation techniques; see, e.g., [2].

Theorem 4: Suppose that (1) is MSS and AWSS and that \( \{w_k\}_{k \in \mathbb{N}_0} \) is white with \( \sigma_n^2 = \text{tr}R_w(0). \) Define

\[ \mathcal{F}(z) = \frac{1}{z - A(p)} - 1 \]
and

\[ \mathcal{C}(p) = \frac{\sigma_n^2/m}{B_w B_n^T + (\sigma_n^2/N)(1-p)\mathcal{E}} \in \mathbb{R}^{(n+N) \times (n+N)} \]
where \( \mathcal{E} \) is defined by [1, Sec.2] for definitions

\[ \mathcal{E} \triangleq \begin{bmatrix} B_w(p)B_n^T & (1-p) \end{bmatrix} \]

Then, the spectral density of \( \{\theta_k\}_{k \in \mathbb{N}_0} \) is given by

\[ S_\theta(s^\omega) = \mathcal{F}(s^\omega)\left[(1-p)\bar{A}R_\theta(0),\bar{A} + \mathcal{C}(p)\right] \]

where \( R_\theta(0) \) solves the following linear matrix equation:

\[ R_\theta(0) = A(p)R_\theta(0)A(p)^T + (1-p)\bar{A}R_\theta(0)\bar{A}^T + C(p). \]

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Proof: See the appendix.

To further elucidate the situation, we note that (5) is linear and that its solution can be stated as the linear combination

\[ \bar{A}(0) = \frac{\sigma_n^2}{m} \bar{R}_w(0) + \left(\frac{\sigma_n^2}{N}\right) \bar{R}_n(0), \]

where \( \bar{R}_w(0) \) and \( \bar{R}_n(0) \) satisfy

\[ \bar{R}_w(0) = A(p)\bar{R}_w(0)A(p)^T + p(1-p)\bar{A}R_w(0)\bar{A}^T + B_w B_n^T \]
and

\[ \bar{R}_n(0) = A(p)\bar{R}_n(0)A(p)^T + p(1-p)\bar{A}R_n(0)\bar{A}^T + (1-p)\mathcal{E}. \]

Therefore, the distortion \( D \) defined by (52) in [1] is given by

\[ D \triangleq \text{tr}(\bar{Q}R_w(0)) + \lambda[0 \ e_1^T]R_w(0)[0 \ e_1^T]^T, \]

where \( \bar{Q} \) is given in terms of the Kronecker product

\[ \bar{Q} \triangleq \begin{bmatrix} [0] & [Q] \\ [0] & [0] \end{bmatrix}. \]

Thus, \( D = \alpha \sigma_n^2 + \beta, \) with

\[ \alpha \triangleq (1/N)\text{tr}(\bar{Q}R_n(0)) + \lambda(0 \ e_1^T)R_w(0)[0 \ e_1^T]^T \]
and

\[ \beta \triangleq (\sigma_n^2/m)\text{tr}(\bar{Q}R_n(0)) + \lambda(0 \ e_1^T)R_w(0)[0 \ e_1^T]^T. \]

The upper expressions replace Lemma 11 of [1].

To derive a noise-shaping model, (6) can be substituted into into (4) to provide

\[ S_\theta(e^{j\omega}) = \mathcal{F}(e^{j\omega})(\sigma_n^2/mK_w K_n^T + (\sigma_n^2/N)K_n K_n^T) e^{-j\omega}. \]

where \( K_w = B_w B_n^T + p(1-p)\bar{A}R_w(0)\bar{A}^T \)
and

\[ K_n = (1-p)(\mathcal{E} + p\bar{A}R_n(0)\bar{A}^T). \]

If we define

\[ \mathcal{H}(z) \triangleq \begin{bmatrix} I & 0 \end{bmatrix} \mathcal{F}(z), \]

then the above provides the noise-shaping model depicted in Fig. 2.

The latter replaces Fig. 2 and Corollary 1 of [1].

Remark 1: We would like to emphasize that Theorem 4 can also be proven by adapting results in [3]–[5]. However, the noise shaping interpretation in Fig. 2 does not explicitly need an additional noise term to quantify second-order dropout effects, as opposed to what is done in [3]–[5].

The upper bound on the coding rate provided by Theorem 5 in [1] is also no longer correct, since it relied upon \( R_\theta(0) \). The new Theorem 5 is provided below:

Theorem 5: For any \( 1 \leq N \in \mathbb{N} \), the minimum bit-rate \( R \) of \( \bar{u}_k \) satisfies

\[ R(D) \leq \frac{1}{2} \log_2 \left( \text{det}(I + (N/\sigma_n^2)R_\xi(0)) \right) + \frac{N}{2} \log_2 \left( \frac{\pi e}{6} \right) + 1, \]

where

\[ R_\xi(0) = \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix} R_\theta(0) \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix}^T. \]

Proof: Follows immediately from (73) in [1] by omitting the last step where \( R_\xi(0) \) was written in terms of \( R_w(0) \) and (50) was used.
By using results in [6, Sec.5], the covariance matrix

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Remark 2: By using results in [6, Sec.5], the covariance matrix

and let \( c \in \mathbb{R}^{(n+N)^2} \) be the vectorized version of the matrix \( C(p) \)
given in (2). Then, the vectorized version of \( R_0(0) \) is simply given by
\( r = (I - G)^{-1} c \). Using this approach, it is straightforward to
numerically evaluate the rate and distortion in (7).

We finalize this note by revisiting the NCS considered in Section V-
C of [1]. Fig. 3 illustrates the rate and distortion trade-off for different
horizon lengths and a fixed packet loss probability \( p = 0.0085 \). It
may be noticed that the distortion can be reduced by using a longer
horizon length in addition to increasing the bit-rate. Fig. 4 shows that
when the packet-loss probability increases, it is necessary to use a
larger horizon length to guarantee stability and thereby reduce the
distortion.

References

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Appendix

Proof of Theorem 4

Since \( \{\nu_k\}_{k\in\mathbb{N}_0} \) is white and thus \( \mathbb{E}\{\theta_k\theta_k^T\} = 0 \), the system
recursion (1) provides

\[
\mathbb{E}\{\theta_{k+1}\theta_{k+1}^T\} = \mathbb{E}\{\hat{A}(d_k)\theta_k\theta_k^T\hat{A}(d_k)^T\} + \mathbb{E}\{\hat{B}(d_k)\nu_k\nu_k^T\hat{B}(d_k)^T\}.
\]

Therefore, by conditioning on \( d_k \) and using the law of total expectation,
we obtain:

\[
\mathbb{E}\{\theta_{k+1}\theta_{k+1}^T\} = p\mathbb{E}\{\hat{A}(d_k)\theta_k\theta_k^T\hat{A}(d_k)^T\} + (1 - p)\mathbb{E}\{\hat{A}(0)\theta_k\theta_k^T\hat{A}(0)^T\}
\]

\[
+ p\mathbb{E}\{\hat{B}(d_k)\nu_k\nu_k^T\hat{B}(d_k)^T\} \quad | \quad d_k = 1
\]

\[
+ (1 - p)\mathbb{E}\{\hat{B}(0)\nu_k\nu_k^T\hat{B}(0)^T\} \quad | \quad d_k = 0
\]

\[
= p\hat{A}(1)\mathbb{E}\{\theta_k\theta_k^T\}\hat{A}(1)^T + (1 - p)\hat{A}(0)\mathbb{E}\{\theta_k\theta_k^T\}\hat{A}(0)^T
\]

\[
+ p\hat{B}(1)\mathbb{E}\{\nu_k\nu_k^T\}\hat{B}(1)^T + (1 - p)\hat{B}(0)\mathbb{E}\{\nu_k\nu_k^T\}\hat{B}(0)^T,
\]

where we have used the fact that \( \{d_k\}_{k\in\mathbb{N}_0} \) is Bernoulli and \( \nu_k \) and
\( \theta_k \) are independent of \( d_k \). Direct algebraic manipulations allow us to
rewrite the above as
\[ E\{\theta_{k+1}^T\theta_{k+1}\} = A(p)E\{\theta_k^T\theta_k\}A(p)^T \]
\[ + p(1-p)\tilde{A}E\{\theta_k^T\theta_k\}\tilde{A}^T + C(p). \]
(8)

In a similar way, one can derive that

\[ E\{\theta_{k+\ell+1}^T\theta_{k+\ell}^T\} = E\{(\tilde{A}(d_{k+\ell})\theta_{k+\ell} + \tilde{B}(d_{k+\ell})\nu_{k+\ell})\theta_{k+\ell}^T\} \]
\[ = E\{\tilde{A}(d_{k+\ell})\theta_{k+\ell}^T\} + E\{\tilde{B}(d_{k+\ell})\nu_{k+\ell}\theta_{k+\ell}^T\} \]
\[ = A(p)E\{\theta_{k+\ell}^T\} + B(p)E\{\nu_{k+\ell}\theta_{k+\ell}^T\} \]
\[ = A(p)E\{\theta_{k+\ell}^T\}, \quad \forall \ell \in \mathbb{N}_0, \]

since \( \{\nu_k\}_{k \in \mathbb{N}_0} \) is white and \( \theta_k \) and \( \theta_{k+\ell} \) are independent of \( d_{k+\ell} \) for non-negative values of \( \ell \).

Equation (9) gives the explicit expression
\[ E\{\theta_{k+\ell}^T\theta_{k+\ell}^T\} = A(p)E\{\theta_k^T\theta_k\}, \quad \forall \ell \in \mathbb{N}_0. \]
(10)

Since the system is AWSS, we have \( \lim_{k \to -\infty} E\{\theta_{k+1}\theta_{k+1}^T\} = R_0(0) \), the stationary covariance matrix of \( \{\theta_k\}_{k \in \mathbb{N}_0} \). By (8) and results in [7], [8], the latter is given by the solution to (5).

On the other hand, in steady state, (10) gives that the covariance function
\[ R_0(\ell) = A(p)^\ell R_0(0), \quad \forall \ell \in \mathbb{N}_0. \]
(11)

Consequently, the positive real part of the spectrum of \( \{\theta_k\}_{k \in \mathbb{N}_0} \) is given by
\[ S_\theta^+(z) = \frac{1}{2} R_0(0) + \sum_{\ell=1}^{\infty} R_0(\ell)z^{-\ell} \]
\[ = (1/2)I + A(p)(zI - A(p))^{-1}R_0(0), \]
where we have used the fact that, by assumption, (1) is MSS and AWSS, thus \( A(p) \) is Schur (see Lemma 4 in [1]) and the geometric series
\[ \sum_{n=0}^{\infty} (A(p)z^{-1})^n = (I - A(p)z^{-1})^{-1}. \]

Since \( \{\theta_k\}_{k \in \mathbb{N}_0} \) is AWSS, its spectrum satisfies [9]
\[ S_\theta(z) = S_\theta^+(z) + (S_\theta^+(z^{-1}))^T \]
\[ = R_0(0) + A(p)(zI - A(p))^{-1}R_0(0) \]
\[ + R_0(0)(z^{-1}I - A(p))^{-T}A(p)^TA(p)^T. \]

Therefore, we have
\[ (zI - A(p))S_\theta(z)(z^{-1}I - A(p))^T \]
\[ = (zI - A(p))R_0(0)(z^{-1}I - A(p))^T \]
\[ + (zI - A(p))A(p)(zI - A(p))^{-1}R_0(0)(z^{-1}I - A(p))^T \]
\[ + (zI - A(p))R_0(0)(z^{-1}I - A(p))^{-T}A(p)^T(z^{-1}I - A(p))^T \]
\[ = (zI - A(p))R_0(0)(z^{-1}I - A(p))^T \]
\[ + A(p)R_0(0)(z^{-1}I - A(p))^T + (zI - A(p))R_0(0)A(p)^T, \]
\[ \text{since } (zI - A(p))A(p)(zI - A(p))^{-1} = A(p). \]

Thus,
\[ \mathcal{F}^{-1}(z)S_\theta(z)\mathcal{F}^{-T}(z^{-1}) \]
\[ = (zR_0(0) - A(p)R_0(0))(z^{-1}I - A(p))^T + z^{-1}A(p)R_0(0) \]
\[ - A(p)R_0(0)A(p)^T + zR_0(0)A(p)^T - A(p)R_0(0)A(p)^T \]
\[ = R_0(0) - z^{-1}A(p)R_0(0) - zR_0(0)A(p)^T \]
\[ + A(p)R_0(0)A(p)^T + z^{-1}A(p)R_0(0) - A(p)R_0(0)A(p)^T \]
\[ + zR_0(0)A(p)^T - A(p)R_0(0)A(p)^T \]
\[ = R_0(0) - A(p)R_0(0)A(p)^T, \]
and (5) establishes (4).