Diffraction of Cylindrical and Plane Waves by an Array of Absorbing Half-Screens

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Abstract—Multiple forward diffraction past an array of many absorbing half-screens whose separation is large compared to wavelength is examined. Starting with the physical optics approximation for half-planes that are equally spaced and of equal height, the field incident on successive edges is represented by a multidimensional Fresnel integral, which is then expanded into a series of functions studied by Boersma. When the angle of incidence with respect to the plane containing the edges is small, as is of interest here, each edge is in the transition region of the previous edge, which precludes the use of the geometrical theory of diffraction and related asymptotic theories. The solution obtained here applies for incidence either from above or below the plane containing the edges, and is especially suited to the case of near-grazing incidence. This method of solution allows for numerical evaluation of a large number of half-screens and shows how the multiply diffracted fields are influenced by the physical parameters. Both incident plane waves and incident cylindrical waves can be treated, and it is found that the numerical results for the cylindrical wave case can be related to those of the plane wave via the settling concept.

I. INTRODUCTION

THIS paper treats the problem of multiple forward diffraction past an array of equally spaced, parallel, absorbing half-screens of uniform height for plane and cylindrical waves incident at small glancing angles from above or below the plane containing the edges. The study is motivated by a desire to predict the propagation of UHF signals in cities for cellular mobile radio and personal communication networks. In the future, such systems will use antennas that are near or even below the diffracting rooftops.

Wallisch and Bertoni [1] have argued that outside of the high rise urban core, the buildings in a city are of nearly uniform height and are organized by the street system into rows, with the space between neighbors in each row being less than the width of the individual buildings. Their theoretical model for propagation from an elevated base station antenna in this environment involves diffraction past many rows of buildings with a final diffraction of the rooftop field down to street level that takes place at the row just before the mobile. To compute the range dependence, the rows of buildings are approximated as absorbing half-screens, so that the propagation over the buildings may be treated as a process of multiple forward diffraction past many half-screens.

Multiple forward diffraction past absorbing half-screens was previously studied in connection with propagation over terrain obstacles, such as mountain ridges and hills, which are well separated and of irregular height. For these conditions, an empirical method [2], in which the terrain profile is replaced by a single knife-edge, is frequently used. However, the method is not satisfactory for two or more nearby hills of relatively uniform height. Although the problem of diffraction past an arbitrary sequence of smooth rounded obstacles has been formulated in terms of an integral equation by Furutsu [3], the formulation does not lend itself to numerical evaluation except for few special cases. Explicit solution for double knife-edge diffraction was given by Millington et al. [4] in terms of a double Fresnel integral that was evaluated using geometrical arguments. For an arbitrary number of knife-edges, Lee [5] achieved analytical solutions for the special case when source, receiver, and edges all lie in a common plane by using path integral methods. From Furutsu's generalized residue series, Vogler [6] deduced a multiple knife-edge attenuation function expressed as repeated integrals of error functions. The evaluation of this representation, which involves an $N$-fold infinite summation, is limited to a few knife-edges since the complexity increases dramatically with the edge number $N$. Whitleker [7] attempted to give a ray representation for multiple knife-edge diffraction in terms of a single modified line source above the preceding diffracting edge. However, this simple ray model is only valid for a few edges since the multiply diffracted field has rapid spatial variations and is not locally that of a cylindrical wave.

For small glancing angles the diffraction problem is complicated by the fact that each edge is in the transition region of the previous edge. As a result, the geometrical theory of diffraction (GTD) cannot be used to describe the process. When dealing with many half-screens, uniform asymptotic theories are too complex to implement, so that one must resort to the physical optics (PO) approximation, which is accurate in the transition region. Wallisch and Bertoni [1] were able to evaluate the diffraction for the case of an incident plane wave by using repeated Kirchhoff–Huygens integrations for each half-screen. Because the integrations were implemented numerically, they were very time consum-
ing and were only made for a limited number of screens, and only for positive angles of incidence. Moreover, they had to make a fundamental assumption to go from the plane wave results to case of fields radiated by a localized antenna (point source).

In this paper we overcome the limitations of Walfisch and Bertoni [1]. Using physical optics approximations, the field incident on successive edges (rooftops) is expressed in terms of a sum over functions studied by Boersma [8]. Incidence may be either from above or below the plane containing the edges, and the solution is especially suited to the case when the angle of incidence with respect to the horizontal is small. This method of solution allows for numerical evaluation for both incident plane waves and cylindrical waves, even when the number of edges is as high as 1000. We show how the multiply diffracted fields are influenced by the angle of incidence, separation distance, frequency and edge number. In the case of incidence from above the plane of the edges, we show that the dependence of the field for an incident plane wave exhibits a settling behavior for a large number of half-screens. Moreover, it is shown that this settling behavior can be used to predict the edge fields for an incident cylindrical wave.

II. MULTIPLE KNIFE-EDGE DIFFRACTION

The series of parallel absorbing half-screens is shown in Fig. 1. The screens lie in the planes \( x = (n - 1)d \) for \( n \geq 1 \), where the separation \( d \) between screens is assumed to be large compared to wavelength \( \lambda \). The edges of all half-screens lie in the plane \( y = 0 \), and for convenience a vertical axis labeled \( y_n \) is associated with each screen. The origin \( x = 0 \) coincides with the edge of the first screen, and a magnetic current line source parallel to the \( z \)-axis, and located at \( x = -d_0, y = y_0 \), radiates a magnetic field having only a \( z \) component. For harmonic time dependence \( \exp(-i\omega t) \), and assuming \( d_0 \gg \lambda \), the phasor field in the plane \( x = 0 \) can be approximated as

\[
H(y) = \frac{e^{ikr_0}}{\sqrt{k}r_0} \quad (1)
\]

where

\[
r_0 = \sqrt{d_0^2 + (y - y_0)^2}.
\]

The field obtained when this cylindrical wave impinges on the array of \( N \) parallel absorbing half-planes will first be derived. The case of an incident plane wave can then be treated in an analogous manner.

A. Physical Optics Approximation

The field of (1) is assumed to be incident on the plane of the \( n = 1 \) half-screen. In order to find the field incident on subsequent half-screens, we make use of the following recursive approach. The field incident on the plane of the \( n + 1 \) absorbing half-screen is found in terms of the field incident on the plane of the \( n \)th half-screen, as suggested in Fig. 1. Using the PO approximation, the equivalent magnetic current distribution on the \( n \)th plane is the same as that induced by the incident field on a perfectly conducting sheet that closes off the aperture in the \( n \)th half-plane. This equivalent current is given by [9]

\[
M_c(y_n) = -\frac{2k}{\omega \delta} \frac{\partial H(y_n)}{\partial x} \quad (3)
\]

for \( y_n \geq 0 \), and is zero for \( y_n < 0 \). Here \( H(y_n) \) is the magnetic field, which is oriented along \( z \) in the plane of the \( n \)th screen.

Using the equivalent magnetic current in the aperture of the plane \( x_n \), the field incident on the plane \( x_{n+1} \) can be calculated by integrating the product of \( M_c \) and the corresponding two-dimensional Green’s function [10], so that

\[
H(y_{n+1}) = i\omega e^{ikR} = \int_0^\infty M_c(y_n)G(R)dy_n \quad (4)
\]

where

\[
R = \sqrt{d^2 + (y_{n+1} - y_n)^2}.
\]

The two-dimensional free space Green’s function \( G(R) \) is proportional to the Hankel function of the first kind. However, since we have assumed that \( d \gg \lambda \), the Hankel function can be approximated by its asymptotic expression, so that

\[
G(R) \approx \frac{1}{4} \sqrt{\frac{2}{\pi kR}} e^{ikR - ikR/4}. \quad (6)
\]

Substituting (3) and (6) into (4), one obtains

\[
H(y_{n+1}) = e^{-ikR/4} \int_0^\infty \frac{1}{\sqrt{k}} \frac{\partial H(y_n)}{\partial x} e^{ikR} dy_n. \quad (7)
\]

The validity of this approximation for treating diffraction through small angles has been discussed in [10].

For small glancing angles, the principle contribution to the integral for a particular value of \( y_{n+1} \) comes from a limited region of \( y_n \) about the value of \( y_{n+1} \). This allows us to use the Fresnel quadratic approximation,

\[
R = d + \frac{(y_{n+1} - y_n)^2}{2d}. \quad (8)
\]

Because propagation is primarily along \( x \) axis, we may use the approximation

\[
\frac{\partial}{\partial x} H(y_n) = ikH(y_n). \quad (9)
\]
Furthermore, the quantity $R$ in the denominator will not differ significantly from $d$. With these approximations (7) becomes

$$
H(y_{n+1}) = \frac{e^{-i\pi/4}}{\sqrt{kd}} e^{i k d} \int_{0}^{\infty} H(y_{n}) e^{i k (y_{n+1} - y_{0})/2 d} dy_{n}.
$$
(10)

In order to derive the field diffracted past a series of $N$ absorbing half-planes, we begin by using the Fresnel approximation (8) for the incident cylindrical wave, and make repeated use of (10). Thus, the field at the receiving point $y_{N+1}$ in the plane of the $N+1$ screen is found to be

$$
H(y_{N+1}) = \frac{e^{-i N \pi/4}}{\sqrt{kd} (\sqrt{\pi})^{N/2}} \exp \left\{ i k (d_{0} + N d) \right\}
$$
$$
\times \int_{0}^{\infty} dy_{1} \int_{0}^{\infty} dy_{2} \cdots \int_{0}^{\infty} dy_{N}
$$
$$
\cdot \exp \left\{ \frac{i k}{2} \left[ (y_{1} - y_{0})^{2}/2 d_{0} + \sum_{n=1}^{N-1} (y_{n+1} - y_{n})^{2}/2 d \right] \right\},
$$
(11)

When $N = 1$, (11) reduces to a Fresnel integral, while for $N = 2$, our result is in agreement with that previously derived by Millington et al. [4] for two absorbing screens. A similar expression was derived by Vogler [6] for half-screens of different heights by reducing from Furutsu's generalized residue series [3] for the propagation of radio waves diffracted by an arbitrary sequence of smooth rounded obstacles to the case of vanishing radii. However, his derivation is not tractable and immediate. Using the path integral method, for the special case of $d_{0} = d$ and $y_{0} = y_{N+1} = 0$, this expression has been also obtained by Lee [5], who further developed some very useful closed-form results.

B. Representation via Boersma’s Functions

In order to proceed with the evaluation of (11) for cylindrical wave incidence it is necessary to assume that $d_{0} = d$ and that $y_{N+1} = 0$. However, unlike Lee [5] we do not require that $y_{0} = 0$. For application to the problem of UHF propagation in cities, these assumptions are not restrictive. Generally, the source is located above one row of buildings, and the field at street level may be found from the field incident on the rooftop ($y_{N+1} = 0$) of the preceding row of buildings using GTD. Further simplification is obtained by defining the new variables

$$
\nu_{n} = \sqrt{-i \frac{k}{2 d}} y_{n}, \quad n = 1, 2, 3, \cdots N.
$$
(12)

With the foregoing, (11) reduces to

$$
H_{N+1} = \frac{e^{ik[(N+1)d + y_{0}]/2 d]} \sqrt{kd} (\sqrt{\pi})^{N/2}} \int_{0}^{\infty} dv_{1} \int_{0}^{\infty} dv_{2} \cdots \int_{0}^{\infty} dv_{N}
$$
$$
\cdot \exp \left\{ \left[ 2 \sqrt{-i \pi g_{c} v_{1}} \right] \right\}
$$
$$
\cdot \exp \left\{ -\nu_{1}^{2} - 2 \sum_{n=2}^{N} \nu_{n}^{2} + 2 \sum_{n=1}^{N-1} \nu_{n} \nu_{n+1} \right\},
$$
(13)

where

$$
g_{c} = \frac{y_{0}}{\sqrt{kd}}.
$$
(14)

It is interesting to note that, except for the phase factor in front of the integrals, the multiple integral depends on the frequency and geometry via the single parameter $g_{c}$, and the frequency and geometry dependence is separate from the dependence on the edge number $N$.

A corresponding expression can be derived for a plane wave incident at a glancing angle $\theta$. Replacing the incident field (1) of a cylindrical wave by a plane wave field $\exp(-i k y_{1} \sin \theta)$ at $x = 0$, the field incident on the edge $y_{N+1} = 0$ of the $N+1$ half-screen is found to be

$$
H_{N+1} = \frac{e^{ikNd}}{(\sqrt{\pi})^{N/2}} \int_{0}^{\infty} dv_{1} \int_{0}^{\infty} dv_{2} \cdots \int_{0}^{\infty} dv_{N}
$$
$$
\cdot \exp \left\{ \left[ 2 \sqrt{-i \pi g_{p} v_{1}} \right] \right\}
$$
$$
\cdot \exp \left\{ -\nu_{1}^{2} - 2 \sum_{n=2}^{N} \nu_{n}^{2} + 2 \sum_{n=1}^{N-1} \nu_{n} \nu_{n+1} \right\},
$$
(15)

where

$$
g_{p} = \sqrt{\frac{\lambda}{d}} \sin \theta.
$$
(16)

Again, the frequency and geometry dependence is contained in a single parameter $g_{p}$ and is separated from the edge number $N$.

We now make Taylor expansions of the exponential $\exp(2 \sqrt{-i \pi g_{p} v_{1}})$ in (15) and the exponential $\exp(2 \sqrt{-i \pi g_{c} v_{1}})$ in (13) around $v_{1} = 0$. With these expansions, we obtain for an incident plane wave

$$
H_{N+1} = e^{ikNd} \sum_{q=0}^{\infty} \frac{[2 \sqrt{-i \pi g_{p}}]^{q}}{q!} I_{N,q}(1),
$$
(17)

while for an incident cylindrical wave

$$
H_{N+1} = e^{i k[(N+1)d + y_{0}]/2 d]} \sum_{q=0}^{\infty} \frac{[2 \sqrt{-i \pi g_{c}}]^{q}}{q!} I_{N,q}(2),
$$
(18)

Here,

$$
I_{N,q}(\alpha) = \frac{1}{(\sqrt{\pi})^{N/2}} \int_{0}^{\infty} dv_{1} \int_{0}^{\infty} dv_{2} \cdots \int_{0}^{\infty} dv_{N}
$$
$$
\times \nu_{1}^{q} \exp \left\{ -\alpha \nu_{1}^{2} - 2 \sum_{n=2}^{N} \nu_{n}^{2} + 2 \sum_{n=1}^{N-1} \nu_{n} \nu_{n+1} \right\},
$$
$$
N = 1, 2, 3, \cdots, q = 0, 1, 2, \cdots.
$$
(19)

Evaluation of this multiple integral is itself an interesting mathematical problem and has been studied by Boersma [8],
who explicitly evaluated $I_{N,q}(1)$ and $I_{N,q}(2)$ for $q = 0, 1$ and gave a recursion relation for $q \geq 1$. For $\alpha = 1$, he found that,

$$I_{N,0}(1) = \frac{(1/2)_N}{N!}$$

(20)

$$I_{N,1}(1) = \frac{1}{2\pi^{1/2}} \sum_{n=0}^{N-1} \frac{(1/2)_n}{n!(N-n-1)!}.$$  (21)

Here $(a)_n$ denotes Pochhammer's symbol defined by $(a)_0 = 1, \quad (a)_n = a(a+1) \cdots (a+n-1), \quad n = 1, 2, 3, \ldots$.  

Also, for $\alpha = 2$, he found that,

$$I_{N,0}(2) = \frac{1}{(N + 1)^{3/2}}$$

(23)

$$I_{N,1}(2) = \frac{1}{4\pi^{1/2}} \sum_{n=1}^{N} \frac{1}{n^{3/2}(N + n - 1)!}.$$  (24)

These functions satisfy the recursion relation

$$I_{N,q}(\alpha) = \frac{N(q - 1)}{2(N + 1)^{\alpha-1}} I_{N,q-1}(\alpha)$$

$$+ \frac{1}{2\pi^{1/2}(N + 1)^{\alpha-1}} \sum_{n=\alpha-1}^{N-1} \frac{I_{n,q-1}(\alpha)}{(N - n)^{3/2}}.$$  (25)

valid for $q \geq 2$, where $(q - 1)I_{N,q-1}(\alpha) = 0$ for $q = 1$, and $I_{0,q}(\alpha) = \delta_{\alpha 0}$ (Kronecker's symbol).

When $\theta = 0^\circ$, so that the plane wave is propagating along $x$, only the zero-order term contributes in (17). Therefore, we obtain the closed-form solution

$$H_{N+1} = e^{i\kappa Nd} \frac{1}{N!}.$$  (26)

Similarly, when $\gamma_0 = 0$, the result for cylindrical wave incidence (18) reduces to

$$H_{N+1} = \frac{e^{i\kappa(N+1)d}}{\sqrt{\kappa(N + 1)d}} \frac{1}{N + 1}.$$  (27)

These closed-form solutions are in agreement with those previously derived by Lee [5] using a path integral method. One important feature of these solutions is that the field amplitude depends only on the edge number $N$, but is independent of frequency and screen separation. This result is analogous to that for a single edge, in which the scattered field is always one-half of the incident field along the shadow boundary.

Once the source is located off the $x$-axis, the fields incident on each edge are obviously dependent on the frequency and geometry as well as edge number through the infinite series of Boersma's functions $I_{N,q}(\alpha)$. The convergent of these series depends on the frequency, separation distance, source height and the edge number. For low angle incidence ($g_0$ or $g_c$ small), the series are rapidly convergent and numerical evaluations have been achieved for up to 1000 screens. A qualitative analysis of the behavior of (17) and (18) is given on the following section.

III. GEOMETRY AND FREQUENCY DEPENDENCE

In the previous section, expressions were obtained for the field incident on successive edges due either to a plane wave or a cylindrical wave. Except for a free space propagation factor before the summations in (17) and (18), these expressions depend separately on the screen number $N$ and on the parameters $g_0$ and $g_c$ that incorporate the geometrical factors and frequency. The parameter grouping of $g_0$ was observed by Walisch and Bertoni [1] working with numerical results for an incident plane wave, and is confirmed here analytically. The edge number dependence is implied in the Boersma's functions $I_{N,0}(1)$ and $I_{N,2}(2)$ for plane wave and cylindrical wave incidence, respectively, and will be discussed for incidence both from below and above the plane of edges ($\gamma = 0$), as well as glancing incidence.

A. Glancing Incidence

When a wave is incident normally on the array so that $\theta = 0^\circ$ in (17) or $\gamma_0 = 0$ in (18), the field reaching each edge depends on $N$, but not on $\theta$ or $\lambda$. In these cases, the fields incident on successive edges reduce to simple closed-form results (26) or (27), and are seen to decrease monotonically with the edge number $N$. Note that the closed-form result (26) or (27) of the multiply diffracted field is simply the product of the corresponding free space field and a factor incorporating the effects due to diffraction by edges. In order to clarify the difference between the plane wave and cylindrical wave results, consider the ratios of the multiply diffracted field at the observation point ($x = Nd$, $y = 0$) to the free space fields that would reach the same point in the absence of the edges at the same point. These ratios, which are real functions of the edge number $N$, can be drawn from (26) and (27) and are given by

$$D_p = \frac{1}{N!}.$$  \hspace{1cm}  (N = 1, 2, 3, \ldots)  \hspace{1cm}  (28)$$

for plane wave incidence, and

$$D_c = \frac{1}{N + 1}.$$  \hspace{1cm}  (N = 1, 2, 3, \ldots)  \hspace{1cm}  (29)$$

for cylindrical wave incidence. These factors are plotted as a function of $N$ in Fig. 2 on a logarithmic scale. For clarity, continuous curves have been drawn. However only the values of $D_p$ and $D_c$ for integer $N$ have physical significance. It is seen from Fig. 2 that both $D_p$ and $D_c$ begin from one-half, which is the well-known results for single knife-edge diffraction; but $D_c$ decreases more rapidly than $D_p$ as $N$ increases. If both the Pochhammer's symbol and the factorial are expressed as Gamma functions, and noting that $\Gamma(N + 1/2)/\Gamma(N + 1) = 1/N^{1/2}$, then $D_p \sim 1/(\pi N)^{1/2}$ for $N$ large [11].
B. Incidence from Below the Plane $y = 0$

Unlike the case of normal incidence, the field incident on each edge is now a function of frequency, source height and separation between edges when $\theta < 0^\circ$ in (17) or $y_0 < 0$ in (18). The magnitude of the field incident on the $N + 1$ edge is plotted as a function $N$ in Fig. 3 for plane wave incidence and Fig. 4 for cylindrical wave incidence. Again, continuous curves have been drawn for clarity, but only values of the field for integer $N$ have physical meaning. Plots are made for various values of the characteristic parameter $g_p$ or $g_c$.

In the plane wave case we have also listed the value of incident angle $\theta$ for screen separation $d/\lambda = 200$, which is a typical value for UHF propagation in cities. For the cylindrical wave case we have listed the source height $y_0$ for $\lambda = 0.2$ m and $d/\lambda = 200$. A logarithmic scale is used again so as to make the field variation evident. The results for normal incidence are shown as well in both Figs. 3 and 4 for the sake of comparison. As expected, the multiple diffracted fields for incidence from below decrease much more rapidly with edge number. Even for a small negative incident angle, such as $\theta = -0.5^\circ$, the edge field for $N = 300$ is 28 dB smaller then the case of a plane wave at normal incidence. When $\theta$ becomes more negative, the decrease with $N$ becomes more rapid.

It is seen from Fig. 4 that for large $N$ the slope of the curves for different values of $g_c < 0$ are all the same as that for glancing incidence. This fact suggest that one of the early edges is acting like a source of a cylindrical wave that is diffracted past the remaining edges. In the case of an incident plane wave with $g_p < 0$, the slope of the curves in Fig. 3 becomes steeper as $N$ increases, approaching a constant for large $N$. The asymptotic slope for plane wave incidence is more negative than it is for cylindrical wave incidence (Fig. 4).

C. Settling Behavior for Plane Wave Incidence from Above the Plane $y = 0$

For the case of plane wave incidence from above the plane of the edges, the field incident on the $N + 1$ edge is shown in Fig. 5 as a function of $N$ for various values of $g_p > 0$. In contrast to the cases of normal incidence and incidence from below the plane $y = 0$, in which the fields incident on successive edge decrease monotonically with $N$ for all $N$, the field here is found to oscillate with decreasing amplitude about a finite value after passing a number of half-planes. Furthermore, instead of decreasing with $N$, the edge field may increase with the edge number and settle to a value

Fig. 2. Variation with screen number $N$ of the diffraction factors $D_p$ and $D_c$ for plane wave and cylindrical wave incidence respectively. Only values of $D_p$ and $D_c$ for integer values of $N$ have physical significance.

Fig. 3. Variation with $N$ of the field amplitude incident on the $N + 1$ edge due to a plane wave incident from below the plane of the edges for several values of $g_p < 0$ (values listed for the angle $\theta$ correspond to $d = 200 \lambda$). Only values of $|H_{N+1}|$ for integer values of $N$ have physical significance.

Fig. 4. Variation with $N$ of the field amplitude reaching the $N + 1$ edge, normalized to the free space field amplitude $1/\sqrt{\kappa R}$ for a cylindrical wave incident from below the plane of the edges for several values of $g_c < 0$ (values listed for the source height $y_0$ correspond to $d = 200 \lambda$). Only values of $|H_{N+1}|$ for integer values of $N$ have physical significance.

Fig. 5. Variation with $N$ of the field amplitude incident on the $N + 1$ edge due to plane wave incidence from above the plane of the edges for several values of $g_p > 0$ (values listed for the angle $\theta$ correspond to $d = 200 \lambda$). Only values of $|H_{N+1}|$ for integer values of $N$ have physical significance.
greater than that of the incident field (taken here to be unity) for relatively large incident angle, as for example when \( \theta = 2.0^\circ \) for \( d/\lambda = 200 \) \( (g_p = 0.494) \) in Fig. 5. The process of settling to a value greater then unity is due to the coherent addition of the incident and diffracted fields, as is well known for diffraction by a single half-screen. When a plane wave is incident at an even larger angle, eg. \( \theta > 5.0^\circ \), so that the edge for \( N = 2 \) is outside the transition region of that for \( N = 1 \), the fields incident on the edges are found to settle to a value close to unity after passing a single screen. Thus, for large angles of incidence the half-planes do not disturb the plane wave field significantly since the diffraction is weak outside the transition regions. For cellular mobile radio, base station antennas are on the order of 40 m high and the distance to the mobile is in the range 1-20 km, so that \( \theta \) ranges from 0.1\(^\circ\) to 2.5\(^\circ\), and hence \( g_p \) is in the range where strong diffraction effects arise from the presence of many absorbing half-screens.

For small angles of incidence, the fields incident on successive edges initially drops to a minimum and gradually increases again. After achieving a maximum it oscillates with decreasing amplitude about a constant value \( Q \). That is shown, for example, in the curve for \( \theta = 0.5^\circ \) \( (g_p = 0.123) \) in Fig. 5. The settled field \( Q \) is a function of \( \theta \), \( d \), and \( \lambda \) through the characteristic parameter \( g_p \) and increases with increasing \( g_p \). As \( g_p \) becomes smaller, the field must pass more edges before it begins to settle, as seen in Fig. 5. However, provided that \( \theta > 0^\circ \), the edge field can be always expected to settle to a constant. The value of \( Q \) is plotted as a function of the characteristic parameter \( g_p \) in Fig. 6 using logarithmic scales. We have also plotted the logarithmic derivative \( s \) given by

\[
s = \frac{d[\log Q(g_p)]}{d[\log (g_p)]}. \tag{30}
\]

It is seen that \( Q \) varies approximately linearly with \( g_p \) for small \( g_p \). However, for large values of \( g_p \), the logarithmic derivative becomes small, and can even be negative.

The settling phenomena can be understood with reference to the Fresnel zone concept. Previous studies [12] have established the relationship between the perturbation of the fields associated with a ray and the Fresnel zone centered about the ray. Let \( \epsilon \ll 1 \) represents a fractional error, and define for a plane wave the corresponding Fresnel zone centered about the ray to the observation point as having half-width given approximately by \( \sqrt{\lambda s/2\pi\epsilon} \), where \( s \) is the distance from the observation point. With these definitions, it has been shown in [12] that an obstacles located outside the Fresnel zone perturbs the field of the plane wave reaching the observation point by a fraction that is less than \( \epsilon \). Although obstacles outside this zone do not perturb the ray field, they may give rise to additional contributions at the observation point as a result of scattering or diffraction. For the geometry of an array of half-planes, the Fresnel zone associated with the plane wave propagating to an edge is a parabola centered about the ray to the edge and with focus at the edge, as depicted in Fig. 7. The edges inside the parabola will perturb the ray field at the edge in question and therefore affect strongly the edge field, while edges outside the parabola do not significantly perturb the field. Thus the distance \( N_0 d \) necessary to achieve settling is that for which the Fresnel zone to the \( N = N_0 \) edge just clears the \( N = 1 \) edge. Exceeding this distance, the edge field will be mainly affected by a constant number of half-planes whose edges are inside the Fresnel zone. As a result, the edge field settles to a constant after passing \( N_0 \) half-planes.

Considering the first Fresnel zone for which \( \epsilon = 1/2\pi \), \( N_0 \) is then determined by the following relation:

\[
N_0 d \tan \theta = \sqrt{N_0} d\lambda \sec \theta. \tag{31}
\]

Solving for \( N_0 \) gives

\[
N_0 = \text{integer} \left\{ \frac{\lambda}{d\sin^2 \theta} \right\} = \text{integer} \left\{ \frac{1}{g_p^2} \right\}. \tag{32}
\]

It is interesting to note that \( N_0 \) is the inverse square of the characteristic parameter describing the frequency and geometry dependence. The values of the \( N_0 \) for each curve in Fig. 5 are indicated by a vertical bar, and are seen to lie close to the first maximum after which the edge field begins to settle.

D. Cylindrical Wave Incidence From Above the Plane \( y = 0 \)

In order to concentrate on the effects of multiple knife-edge diffraction, we consider the excess attenuation, which is the ratio of the field reaching the \( N + 1 \) edge after multiple
diffraction to the field propagating in free space to the same point when no half-screens are present. The values of excess attenuation for a cylindrical wave incidence from above the \( y = 0 \) plane are shown in Fig. 8 as a function of \( N \) for different values of the parameter \( g_p \). The curves are also parameterized by the value of antenna height \( y_0 \) for \( d/\lambda = 200 \) and \( \lambda = 0.2 \) m. As before, we have drawn continuous curves for clarity, but only values of excess attenuation for integer value of \( N \) have physical meaning. The excess attenuation, as shown in Fig. 8, may be initially greater than unity for large values of \( g_p \) due to the coherence of the incident and edge diffracted fields. After passing the first few edges, the field decreases monotonically with \( N \), although it decreases very slowly as \( N \) becomes large. However, for all values of \( g_p \), the excess attenuation does not show the settling behavior found for plane wave incidence. In contrast to the plane wave case, the angle between the \( x \) axis and the ray from the source to the \( N + 1 \) edge decreases monotonically with \( N \). This angle variation precludes settling in the sense found for plane waves. However, making use of this angle, a relation is found between the field due to an incident cylindrical wave and the settled value \( Q \) for an incident plane wave.

The ray from the source located at \((-d, y_0)\) to the edge of the \( N + 1 \) half-screen makes an angle \( \theta_{N+1} \) with the \( x \)-axis given by

\[
\theta_{N+1} = \tan^{-1} \left( \frac{y_0}{(N + 1)d} \right).
\]

The length \( r_{N+1} \) of the ray is given by

\[
r_{N+1} = \sqrt{y_0^2 + (N + 1)d^2}.
\]

Using these values we have found for small \( \theta_{N+1} \) that the field \( H_{N+1} \) due to an incident cylindrical wave, satisfies the approximation

\[
\sqrt{k r_{N+1}} \mid H_{N+1} \mid = Q(g_p),
\]

when \( g_p = \sqrt{d/\lambda} \sin \theta_{N+1} \). This result is demonstrated in Fig. 9 where we have plotted the excess attenuation \( \sqrt{k r_{N+1}} \mid H_{N+1} \mid \) for cylindrical wave incidence for \( y_0 = 12.5 \) m as a solid curve versus \( N \). For comparison, the values of the settled field \( Q(g_p) \) for plane wave incidence, when \( g_p = \sqrt{d/\lambda} \sin \theta_{N+1} \), are plotted as stars, and are seen to give good agreement with the results for cylindrical wave incidence. It is found that as \( y_0 \) increases the curve of excess attenuation is even closer to the values of settled fields, while the agreement is not as good for smaller values of \( y_0 \).

### IV. IMPLICATIONS FOR DISTANCE DEPENDENCE

As seen from the above discussion for a cylindrical wave incident from the plane of the edges, the excess attenuation \( \sqrt{k r_{N+1}} \mid H_{N+1} \mid \) for the field incident on the \( N + 1 \) edge approaches the settled value \( Q \) of the edge field for plane wave incidence. This implies that we can factor out the distance dependence due to multiple knife-edge diffraction from that due to the free space propagation for an elevated source. Note that

\[
g_p = \sqrt{d/\lambda} \sin \theta = \frac{\sqrt{d/\lambda} y_0}{R}
\]

where \( y_0 \) is the height of the source above the edges and \( R \) is the distance from the source to the edge in question. From the plot of \( Q(g_p) \) and its logarithmic derivative in Fig. 6, it is seen that

\[
Q(g_p) \propto g_p^2 \alpha 1/R^3
\]

and \( Q(g_p) \) varies inversely with \( R \) to a power less than or equal to unity. The total distance dependence can be found by including the free space distance dependence \( 1/\sqrt{R} \) in the case of a line source, or \( 1/R \) in the case of a point source. As \( g_p \to 0 \), the slope approaches unity, so that including the foregoing variation of \( Q \) with \( R \), the field varies as \( 1/R^2 \) for an incident spherical wave. It is well known that for a point source above a dielectric half-space, the field at the interface varies inversely as the square of the distance along the interface for distance large compared to the source elevation [2]. It is surprising to find a similar distance dependence.
for an array of absorbing half-planes whose separation is large compared to wavelength.

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REFERENCES


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