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Lecture 3: Sampling and reconstruction, transform analysis of LTI systems

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Course at a glance
Part I-A: Periodic sampling

- Part I: sampling and reconstruction
  - Periodic sampling
  - Frequency domain representation of the sampling
  - Reconstruction
- Part II: system analysis

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Periodic sampling

- From continuous-time $x_c(t)$ to discrete-time $x[n]$:

$$x[n] = x_c(nT), \quad -\infty < n < \infty$$

- Sampling period $T$
- Sampling frequency $f_s = 1/T$
  $$\Omega_s = 2\pi / T$$

Figure 4.1 Block diagram representation of an ideal continuous-to-discrete-time (C/D) converter.
Two stages

- Mathematically
  - Impulse train modulator
  - Conversion of the impulse train to a sequence

\[
s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)
\]

\[
x_c(t) = x_c(t)s(t)
= x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)
= \sum_{n=-\infty}^{\infty} x_c(nT) \delta(t - nT)
\]

\[
x[n] = x_c(nT), \quad -\infty < n < \infty
\]

\[
(x_c(t) = \int_{-\infty}^{\infty} x_c(\tau) \delta(t - \tau) d\tau)
\]

In practice?

Periodic sampling

- Tow-stage representation
  - Strictly a mathematical representation that is convenient for gaining insight into sampling in both the time and frequency domains.
  - Physical implementation is different.
  - \( x_c(t) \) a continuous-time signal, an impulse train, zero except at \( nT \)
  - \( x[n] \) a discrete-time sequence, time normalization, no explicit information about sampling rate

- Many-to-many \( \rightarrow \) in general not invertible
Part I-B: Freq. domain represent.

- Part I: sampling and reconstruction
  - Periodic sampling
  - Frequency domain representation of the sampling
  - Reconstruction
- Part II: system analysis

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**Frequency-domain representation**

- From $x_c(t)$ to $x_s(t)$  
  The Fourier transform of a periodic impulse train is a periodic impulse train.

\[
s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) \quad \leftrightarrow \quad S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega-k\Omega_s) \\
x_s(t) = x_s(t)s(t) \quad \leftrightarrow \quad X_s(j\Omega) = \frac{1}{2\pi} X_s(j\Omega)^* S(j\Omega) \\
= x_s(t) \sum_{n=-\infty}^{\infty} \delta(t-nT) \quad = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_s(j(\Omega-k\Omega_s)) \\
= \sum_{n=-\infty}^{\infty} x_s(nT) \delta(t-nT)
\]

- The Fourier transform of $x_s(t)$ consists of periodic repetition of the Fourier transform of $x_s(t)$. 

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Recovery

\[ X_r(j\Omega) = H_r(j\Omega)X_s(j\Omega) \]

Ideal lowpass filter with gain T and cutoff frequency \( \Omega_c \)

\[ \Omega_N < \Omega_c < (\Omega_s - \Omega_N) \]

\[ X_r(j\Omega) = X_s(j\Omega) \]

Figure 4.4 Exact recovery of a continuous-time signal from its samples using an ideal lowpass filter.
Aliasing distortion

- Due to the overlap among the copies of $X_c(j\Omega)$, due to $\Omega_s \leq 2\Omega_N$
- $X_c(j\Omega)$ not recoverable by lowpass filtering

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Aliasing – an example

$$X_c(t) = \cos \Omega_0 t$$

$$X_c(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

$$X_r(t) = \cos \Omega_0 t$$

$$x_r(t) = \cos(\Omega_0 - \Omega_0) t$$

Figure 4.5: The effect of aliasing in the sampling of a cosine signal.
Nyquist sampling theorem

Given bandlimited signal $x_c(t)$ with

$$X_c(j\Omega) = 0, \quad \text{for } |\Omega| \geq \Omega_N$$

Then $x_c(t)$ is uniquely determined by its samples

$$x[n] = x_c(nT), \quad -\infty < n < \infty$$

If

$$\Omega_s = \frac{2\pi}{T} \geq 2\Omega_N$$

$\Omega_N$ is called Nyquist frequency

$2\Omega_N$ is called Nyquist rate

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Fourier transform of $x[n]$

From $x_s(t)$ to $x[n]$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT)$$

$$x[n] = x_c(nT), \quad -\infty < n < \infty$$

From $X_s(j\Omega)$ to $X(e^{j\omega})$

By taking continuous-time Fourier transform of $x_s(t)$

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega n} \quad (X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt)$$

By taking discrete-time Fourier transform of $x[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad X_s(j\Omega) = X(e^{j\omega})|_{\omega = \Omega_T} = X(e^{j\Omega_T})$$
Fourier transform of $x[n]$

From Slide 14, $X_s(j\Omega) = X(e^{j\omega})|_{\omega=\Omega T} = X(e^{j\Omega T})$

From Slide 8, $X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_s(j(\Omega - k\Omega_s))$

So, $X(e^{j\omega}) = X_s(j\Omega)|_{\Omega=\omega T} = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_s(j(\Omega - k\Omega_s))|_{\Omega=\omega T}$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_s(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_s\left(j\left(\frac{\omega - 2\pi k}{T}\right)\right)$$

i.e. $X(e^{j\omega})$ is simply a frequency-scaled version of $X_s(j\Omega)$ with $\omega = \Omega T$

$x_s(t)$ retains a spacing between samples equal to the sampling period $T$ while $x[n]$ always has unity space.

Sampling and reconstruction of Sin Signal

$x_s(t) = \cos(4000\pi t) \rightarrow \Omega_s = 4000\pi$

$T = 1/6000 \rightarrow \Omega_T = 2\pi / T = 12000\pi$ \quad no aliasing

$x[n] = x_s(nT) = \cos(4000nT) = \cos((2\pi / 3)n) = \cos(\omega_n n)$

$x_s(t) \leftrightarrow X_s(j\Omega) = \pi \delta(\Omega - 4000\pi) + \pi \delta(\Omega + 4000\pi)$

$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_s(j(\Omega - k\Omega_s))$

$X(e^{j\omega}) = X_s(j\Omega)|_{\Omega=\omega T} = X_s(j\omega / T)$ with normalized frequency $\omega = \Omega T$

How about $x_s(t) = \cos(16000\pi t)$

Figure 4.6 Continuous-time (a) and discrete-time (b) Fourier transforms for sampled cosine signal with frequency $\Omega_s = 4000\pi$ and sampling period $T = 1/6000$. 
Part I-C: Reconstruction

- Part I: sampling and reconstruction
  - Periodic sampling
  - Frequency domain representation of the sampling
  - Reconstruction
- Part II: system analysis

Requirement for reconstruction

- On the basis of the sampling theorem, samples represent the signal exactly when:
  - Bandlimited signal
  - Enough sampling frequency
  - + knowledge of the sampling period \( \rightarrow \) recover the signal
Reconstruction steps

(1) Given \( x[n] \) and \( T \), the impulse train is

\[
x_c(t) = \sum_{n=-\infty}^{\infty} x_c(nT) \delta(t - nT) = \sum_{n=-\infty}^{\infty} x[n] \delta(t - nT)
\]

i.e. the \( n \)th sample is associated with the impulse at \( t=nT \).

(2) The impulse train is filtered by an ideal lowpass CT filter with impulse response \( h_r(t) \leftrightarrow H_r(j\Omega) \)

\[
x_r(t) = \sum_{n=-\infty}^{\infty} x(n) h_r(t - nT)
\]

\[
X_r(j\Omega) = H_r(j\Omega) X(e^{j\Omega T})
\]
Ideal lowpass filter interpolation

CT signal

Modulated impulse train

\[ x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin(\pi(t-nT)/T)}{\pi(t-nT)/T} \]

Figure 4.9 Ideal bandlimited interpolation.

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Ideal discrete-to-continuous-time converter

(a) Ideal reconstruction system

(b) Equivalent representation as an ideal D/C converter.

\[ x[n] \rightarrow x_r(t) \rightarrow x_c(t) \]

Figure 4.10 (a) Ideal bandlimited signal reconstruction. (b) Equivalent representation as an ideal D/C converter.
**discrete-to-continuous-time converter**

“Practical DACs do not output a sequence of dirac impulses (that, if ideally low-pass filtered, result in the original signal before sampling) but instead output a sequence of piecewise constant values or rectangular pulses.”


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**Applications**

![Figure 4.11](image1.png) **Figure 4.11** Discrete-time processing of continuous-time signals.

![Figure 4.16](image2.png) **Figure 4.16** Continuous-time processing of discrete-time signals.
Part II System analysis

- Part I: sampling and reconstruction
- Part II: system analysis
  - Frequency response
  - System functions
  - All-pass systems
  - Minimum-phase systems
  - Linear systems with generalized linear phase

System analysis

- Three domains
  - Time domain: impulse response, convolution sum
    \[ y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] \]
  - Frequency domain: frequency response
    \[ Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) \]
  - z-transform: system function
    \[ Y(z) = X(z)H(z) \]
- LTI system is completed characterized by …
Part II-A: Frequency response

- Part I: sampling and reconstruction
- Part II: system analysis
  - Frequency response
  - System functions
  - All-pass systems
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Frequency response

- Relationship btw Fourier transforms of input and output
  \[ Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) \]
- In polar form
  - Magnitude \( \rightarrow \) magnitude response, gain, distortion
    \[ |Y(e^{j\omega})| = |X(e^{j\omega})| \cdot |H(e^{j\omega})| \]
  - Phase \( \rightarrow \) phase response, phase shift, distortion
    \[ \angle Y(e^{j\omega}) = \angle X(e^{j\omega}) + \angle H(e^{j\omega}) \]
Ideal lowpass filter – an example

- Frequency response
  \[ H(e^{j\omega}) = \begin{cases} 
  1, & |\omega| < \omega_c, \\
  0, & \omega_c < |\omega| < \pi 
\end{cases} \]

- Frequency selective filter

- Impulse response
  \[ h_p[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty \]
  \[ h[n] = 0, \quad n < 0 \]

- Noncausal, cannot be implemented!

- How to make a noncausal system causal?
  - In general, any noncausal FIR system can be made causal by cascading it with a sufficiently long delay!
  - But ideal lowpass filter is an IIR system!

Phase distortion and delay

- Ideal delay system
  \[ h_{id}[n] = \delta[n - n_d] \]
  \[ H_{id}(e^{j\omega}) = e^{-j\omega n_d} \]
  \[ |H_{id}(e^{j\omega})| = 1 \]
  \[ \angle H_{id}(e^{j\omega}) = -\omega n_d, \quad |\omega| < \pi \]

- Ideal lowpass filter with linear phase
  \[ H_p(e^{j\omega}) = \begin{cases} 
  e^{-j\omega n_d}, & |\omega| < \omega_c, \\
  0, & \omega_c < |\omega| < \pi 
\end{cases} \]
  \[ h_p[n] = \frac{\sin \omega_c (n - n_d)}{\pi (n - n_d)}, \quad -\infty < n < \infty \]

Ideal lowpass filter is always noncausal!
Group delay

- A measure of the linearity of the phase
- Concerning the phase distortion on a narrowband signal

\[ x[n] = s[n]\cos(\omega_0 n) \]

- For this input with spectrum only around \( \omega_0 \), phase effect can be approximated around \( \omega_0 \) as the linear approximation (though in reality maybe nonlinear)

\[ \angle H(e^{j\omega}) \approx -\omega n_d - \phi_0 \]

and the output is approximately

\[ y[n] \approx H(e^{j\omega}) \cdot s[n-n_d] \cos(\omega_0(n-n_d) - \phi_0) \]

- Group delay

\[ \tau = \text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega} \text{arg}[H(e^{j\omega})] \]

An example of group delay

![Group delay example](image)
An example of group delay

Since the filter has considerable attenuation at $\omega = 0.85\pi$, the pulse at that frequency is not clearly present in the output. Also, since the group delay at $\omega = 0.25\pi$ is approximately 200 samples and at $\omega = 0.5\pi$ is approximately 50 samples, the second pulse in $x[n]$ will be delayed by about 200 samples and the third pulse by 50 samples, as we see is the case in Figure 5.3.

Part II-B: System functions

- Part I: sampling and reconstruction
- Part II: system analysis
  - Frequency response
  - System functions
  - All-pass systems
  - Minimum-phase systems
  - Linear systems with generalized linear phase
System function of LCCDE systems

- Linear constant-coefficient difference equation
  \[ \sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m] \]

- z-transform format
  \[ \sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{m=0}^{M} b_m z^{-m} X(z) \]
  \[ H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{N} a_k z^{-k} \]
  \[ = \frac{b_0}{a_0} \prod_{n=1}^{M} (1-c_n z^{-1}) \]
  \[ \quad \times \prod_{l=1}^{L} (1-d_l z^{-1}) \]
  (1\(c_m z^{-1}\)) in the numerator
  a zero at \(z = c_m\) a pole at \(z = 0\)
  (1\(d_i z^{-1}\)) in the denominator
  a zero at \(z = 0\) a pole at \(z = d_i\)

Stability and causality

- Stable
  - \(h[n]\) absolutely summable
  - \(H(z)\) has a ROC including the unit circle

- Causal
  - \(h[n]\) right side sequence
  - \(H(z)\) has a ROC being outside the outermost pole
Inverse systems

- Many systems have inverses, specially systems with rational system functions

\[
G(z) = H(z)H_i(z) = 1
\]

\[
H(z) = \frac{b_0}{d_0} \prod_{n=1}^{M} (1 - c_nz^{-1})
\]

\[
H_i(z) = \frac{1}{H(z)}
\]

\[
g[n] = h[n] * h_i[n] = \delta[n]
\]

- Poles become zeros and vice versa.
- ROC: must have overlap btw the two for the sake of G(z).

Example

\[
H(z) = \frac{1 - 0.5z^{-1}}{1 - 0.9z^{-1}}, |z| > 0.9
\]

\[
H_i(z) = \frac{1 - 0.9z^{-1}}{1 - 0.5z^{-1}} = \frac{1}{1 - 0.5z^{-1}} - \frac{0.9z^{-1}}{1 - 0.5z^{-1}}
\]

So,

\[
|z| > 0.5
\]

\[
h_i[n] = (0.5)^n u[n] - 0.9(0.5)^{n-1} u[n - 1]
\]
Part II-C: All-pass systems

- Part I: sampling and reconstruction
- Part II: system analysis
  - Frequency response
  - System functions
  - All-pass systems
  - Minimum-phase systems
  - Linear systems with generalized linear phase

All-pass systems

- Consider the following stable system function

\[ H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}} \]

\[ H_{ap}(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}} \]

\[ = e^{-j\omega} \frac{1 - a^* e^{j\omega}}{1 - ae^{-j\omega}} \]

- \( |H_{ap}(e^{j\omega})| = 1 \) all-pass system: for which the frequency response magnitude is a constant.
Example: First-order all-pass system

P275 Example 5.13

Part II-D: Minimum-phase systems

- Part I: sampling and reconstruction
- Part II: system analysis
  - Frequency response
  - System functions
  - All-pass systems
  - Minimum-phase systems
  - Linear systems with generalized linear phase
Minimum-phase systems

- Magnitude does not uniquely characterize the system
  - Stable and causal $\rightarrow$ poles inside unit circle, no restriction on zeros
  - Zeros are also inside unit circle $\rightarrow$ inverse system is also stable and causal (in many situations, we need inverse systems!)
  - $\rightarrow$ such systems are called minimum-phase systems (explanation to follow): are stable and causal and have stable and causal inverses

Minimum-phase and all-pass decomposition

Any rational system function can be expressed as:

$$H(z) = H_{\min}(z)H_{ap}(z)$$

Suppose $H(z)$ has one zero outside the unit circle at $z = 1/c^*, |c| < 1$

$$H(z) = H_{1}(z)(z^{-1} - c^*)$$

$$= H_{1}(z)(1 - cz^{-1}) \frac{z^{-1} - c^*}{1 - cz^{-1}}$$

**minimum-phase** all-pass
**Frequency response compensation**

When the distortion system is not minimum-phase system:

\[ H_d(z) = H_{d\min}(z)H_{ap}(z) \]

\[ H_c(z) = \frac{1}{H_{d\min}(z)} \]

\[ G(z) = H_d(z)H_c(z) = H_{ap}(z) \]

Frequency response magnitude is compensated

Phase response is the phase of the all-pass

---

**Properties of minimum-phase systems**

- From minimum-phase and all-pass decomposition
  \[ H(z) = H_{\min}(z)H_{ap}(z) \]
  \[ \arg[H(e^{j\omega})] = \arg[H_{\min}(e^{j\omega})] + \arg[H_{ap}(e^{j\omega})] \]

- From the previous figure, the continuous-phase curve of an all-pass system is negative for \( 0 \leq \omega \leq \pi \)

- So change from minimum-phase to non-minimum-phase (+all-pass phase) always decreases the continuous phase or increases the negative of the phase (called the phase-lag function). Minimum-phase is more precisely called minimum phase-lag system
Part II-E: GLP systems

- Part I: sampling and reconstruction
- Part II: system analysis
  - Frequency response
  - System functions
  - All-pass systems
  - Minimum-phase systems
  - Linear systems with generalized linear phase

Design a system with non-zero phase

- System design sometimes desires
  - Constant frequency response magnitude
  - Zero phase, when not possible
    - accept phase distortion, in particular linear phase since it only introduce time shift
    - Nonlinear phase will change the shape of the input signal though having constant magnitude response
Ideal delay

\[ H_{id}(e^{j\omega}) = e^{-j\alpha}, \quad |\omega| < \pi \]

\[ |H_{id}(e^{j\omega})| = 1 \]

\[ \angle H_{id}(e^{j\omega}) = -\omega \alpha, \quad |\omega| < \pi \]

\[ \text{grd}[H_{id}(e^{j\omega})] = \alpha \]

\[ h_{id}[n] = \frac{\sin \pi(n - \alpha)}{\pi(n - \alpha)} \quad \text{Ideal lowpass with linear phase} \]

when \( \alpha = n_d \)

\[ h_{id}[n] = \delta[n - n_d] \]

\[ h_{lp}[n] = \frac{\sin \omega_s(n - n_d)}{\pi(n - n_d)} \]

Generalized linear phase

- Linear phase filters
  \[ H(e^{j\omega}) = |H(e^{j\omega})|e^{-j\alpha} \]

- Generalized linear phase filters
  \[ H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha + j\beta} \]

\( A(e^{j\omega}) \) is a real function of \( \omega \), 
\( \alpha \) and \( \beta \) are real constants
Summary

- Part I: sampling and reconstruction
  - Periodic sampling
  - Frequency domain representation
  - Reconstruction
- Part II: system analysis
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